

A COMPARATIVE ANALYSIS OF ANTENNA BEAM PATTERNS OF THE 6-BY-1 LINEAR ARRAY AND THE 6-BY-2 RECTANGULAR ARRAY

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I. ABSTRACT

An *antenna array* is any collection of isotropic point sources arranged according to a specified geometry. A *linear array* is an antenna array where the antenna elements (i.e. the point sources) are arranged in a straight line. It is a 1-Dimensional array. A *planar array* is an antenna array arranged in a plane in some particular way. It is a 2-Dimensional array. Common examples are rectangular arrays, circular arrays and elliptical arrays.

In this paper we are considering a linear array with 6 elements (called the 6×1 array) and a rectangular array with dimensions 6×2 . Also, for simplicity, the distance between any two elements in any direction is assumed to be constant and equal.

Here we are concentrating on obtaining the beam patterns of these antenna arrays and measuring important parameters like mainlobe beam width (called bore sight) and directivity for different choices of weights. The analytical expressions for

beam patterns are plotted and confirmed independently using MATLAB

II. THE 6×1 LINEAR ARRAY

Consider a uniform linear array with six elements arranged in a straight line along the

x-axis of the coordinate system as shown below:

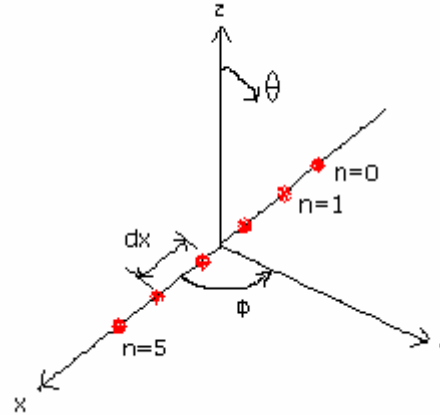


Figure 1

The distance between any two elements is taken to be d_x . The x-coordinate of n^{th} element, x_n is given by

$$x_n = \left(n - \frac{5}{2}\right) d_x, \quad n = 0, 1, \dots, 5 \quad \dots(1)$$

The array manifold vector (denoted in bold-face) is defined by

$$\mathbf{v}_k(k_x) = \left[e^{j\frac{5}{2} k_x d_x} \ e^{j\frac{3}{2} k_x d_x} \ \dots \ e^{-j\frac{3}{2} k_x d_x} \ e^{-j\frac{5}{2} k_x d_x} \right]^T \quad \dots(2)$$

with the n^{th} element being $e^{-j k_x x_n}$ ($n=0, 1 \dots 5$) where

$$k_x = -\frac{2\pi}{\lambda} \sin[\theta] \cos[\phi] = -\frac{2\pi}{\lambda} u_x \quad \dots(3)$$

$$u_x = \sin[\theta] \cos[\phi]$$

Here, k_x is the wavenumber along x-axis, u_x is the unit vector along x-axis in spherical coordinates and λ is the operating wavelength. Obviously, the visible region is $-1 \leq u_x \leq 1$.

The beam pattern in the u-space is given by

$$B_u(u) = \mathbf{w}^H \mathbf{v}_u(u) = e^{-j\frac{5}{2} \frac{2\pi}{\lambda} d_x u_x} \sum_{n=0}^5 w_n^* e^{jn \frac{2\pi}{\lambda} d_x u_x} \dots(4)$$

where the weights of the antenna array are taken to be

$$\mathbf{w} = [w_0 \ w_1 \ w_2 \ w_3 \ w_4 \ w_5] \dots(5)$$

Here, \mathbf{w}^H is the Hermitian conjugate of \mathbf{w}

Similarly the actual beam pattern in 3-D space (i.e. in terms of θ and ϕ) is obtained by substituting the expression for u_x

III. BEAM PATTERNS OF THE 6 × 1 ARRAY

The different choices of \mathbf{w} give a plethora of beam patterns that can be applied practically. Here we shall specifically concentrate on five important weighting functions:

- (a) Uniform weighting
- (b) Hamming weighting
- (c) Hann weighting
- (d) Blackman weighting
- (e) Kaiser weighting

We compare these weighting functions with respect to crucial parameters such as directivity and bore sight (i.e. width of the mainlobe).

The directivity D is given by

$$D = \left\{ \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi |B(\theta, \phi)|^2 \sin \theta \, d\theta \, d\phi \right\}^{-1} \dots(6)$$

Where $|x|$ denotes absolute value of x and $|B(\theta, \phi)|^2$ gives the normalized power pattern as a function of u . When $d_x = \lambda / 2$, we call the array as a *standard linear array*. For such an array, the directivity is

$$D = \left\{ \sum_{n=0}^5 |w_n|^2 \right\}^{-1} \dots(7)$$

Practically speaking, eq (6) has a closed-form expression only in a few cases. Hence in general, we perform numerical integration to obtain directivity.

The boresight depends upon the choice of weighting function, which we shall consider in subsequent sections.

• Uniform Weighting

In uniform weighting,

$$w_n = \frac{1}{6} \text{ for } n=0, 1, \dots, 5 \dots(8)$$

the *normalized* beam pattern is given by

$$B_u(u) = \frac{1}{6} \frac{\text{Sin}(6\pi d_x u_x / \lambda)}{\text{Sin}(\pi d_x u_x / \lambda)} \text{ where } |u_x| \leq 1 \dots(9)$$

For a standard linear array $d_x = \lambda / 2$ hence

$$B_u(u) = \frac{1}{6} \frac{\text{Sin}(3\pi u_x)}{\text{Sin}(\pi u_x / 2)} \text{ where } |u_x| \leq 1 \dots(10)$$

This pattern is plotted using MATLAB in Figure 2 as a function of u_x and in Figure 3 as a function of θ and ϕ .

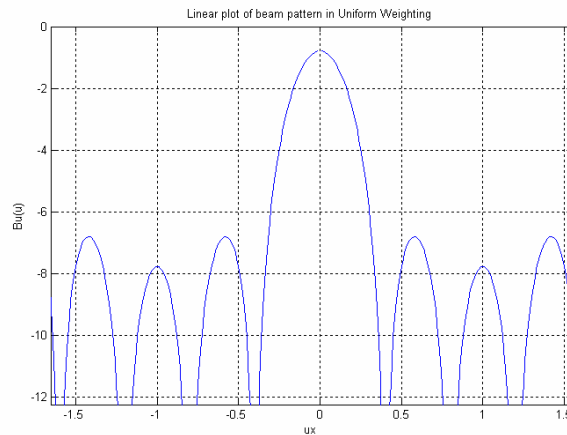


Figure 2

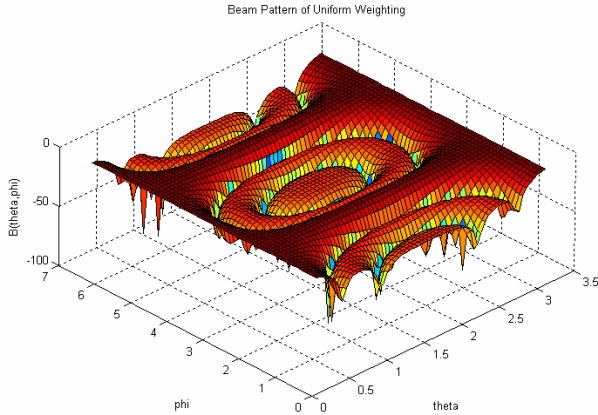


Figure 3

The boresight (BeamWidth between First Null - BWFN) in the u-space is given by:

$$\text{BWFN} = 2\lambda / (6 d_x) \quad \dots(11)$$

which, for a standard linear array becomes,

$$\text{BWFN} = 4 / 6 = 0.666 \quad \dots(12)$$

The directivity D is given by

$$D = 4 / \text{BWFN} \quad \dots(13)$$

which, for a standard linear array becomes,

$$D = 6 \quad \dots(14)$$

The same result is obtained by using eq(8) in eq(7) simplifying the summation.

• Hamming Weighting

In Hamming weighting we place a null at

$$u_x = 3 / 6 = 1 / 2$$

and normalize the response at broadside to unity. Thus,

$$w(n) = 0.54 - 0.46 \cos\left(\frac{2\pi n}{5}\right) \quad \dots(15)$$

for $n=0,1,\dots,5$

Performing the summation (with $d_x = \lambda / 2$) gives

$$B_u(u) = 0.54 \frac{\sin(3\pi u_x)}{\sin(\pi u_x / 2)} - 0.23 \left(\frac{\sin(3\pi(u_x - 1/3))}{\sin(\pi(u_x - 1/3)/2)} + \frac{\sin(3\pi(u_x + 1/3))}{\sin(\pi(u_x + 1/3)/2)} \right) \quad \dots(16)$$

This pattern is plotted using MATLAB in Figure 4 and Figure 5

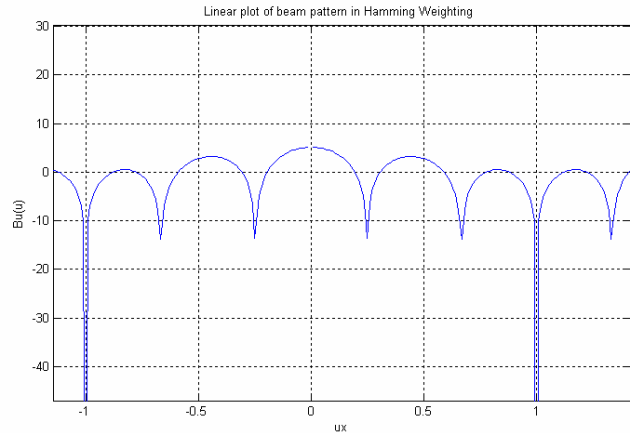


Figure 4

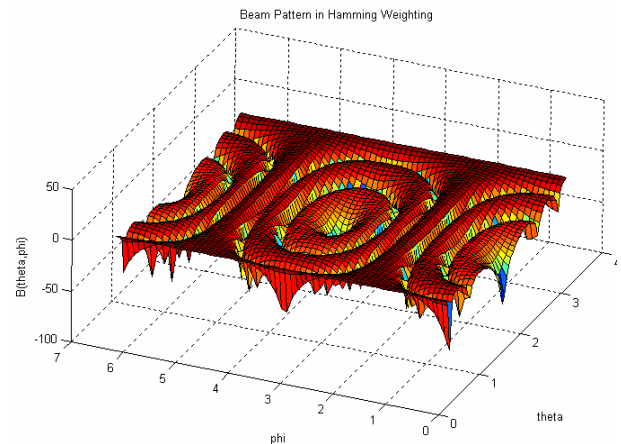


Figure 5

The directivity is obtained by

$$D = 0.664 * 6 = 3.984 \dots \quad \dots(17)$$

which is an improvement over Uniform Weighting. The boresight (in u-space) is

$$\text{BWFN} = 4 * 2 / 6 = 4 / 3 = 1.333 \quad \dots(18)$$

Hence, we see a trade-off here. Reducing the directivity increases BWFN. This is a general trend.

C. Hann Weighting

In Hann weighting, the weights are given by:

$$w(n) = 0.5 - 0.5 \cos\left(\frac{2\pi n}{5}\right) \quad \dots(19)$$

for $n=0,1,\dots,5$

Performing the summation (with $d_x = \lambda / 2$) it gives

$$B_u(u) = 0.5 \frac{\sin(3\pi u_x)}{\sin(\pi u_x / 2)} - 0.25 \left(\frac{\sin(3\pi(u_x - 1/3))}{\sin(\pi(u_x - 1/3)/2)} + \frac{\sin(3\pi(u_x + 1/3))}{\sin(\pi(u_x + 1/3)/2)} \right) \quad \dots(20)$$

This pattern is plotted using MATLAB in Figure 6 and Figure 7

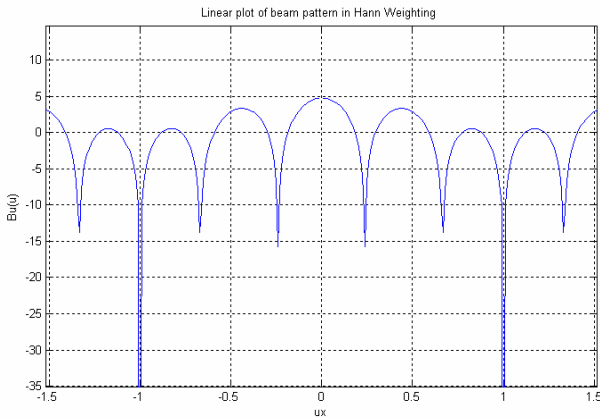


Figure 6

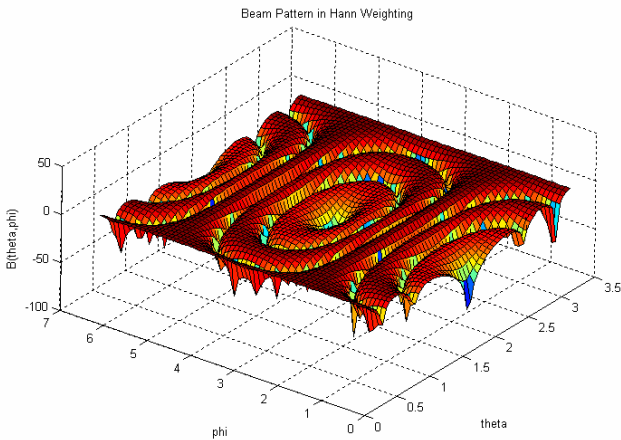


Figure 7

The directivity is given by

$$D = 0.667 * 6 = 4.002 \quad \dots(21)$$

This is an improvement over Uniform Weighting but worse than Hamming Weighting. The boresight (in u-space) is

$$BWFN = 4 * 2 / 6 = 4 / 3 = 1.333 \quad \dots(22)$$

This is same as that of Hamming Weigthing.

D. Blackman Weighting

Blackman proposed the following formula for weights so as to produce nulls at the peaks of first two sidelobes.

$$w(n) = 0.42 - 0.5 \cos\left(\frac{2\pi n}{5}\right) + 0.08 \cos\left(\frac{4\pi n}{5}\right) \quad \dots(23)$$

for $n=0,1,\dots,5$

Performing the summation gives

$$B_u(u) = 0.42 \frac{\sin(3\pi u_x)}{\sin(\pi u_x / 2)} - 0.25 \left(\frac{\sin(3\pi(u_x - 1/3))}{\sin(\pi(u_x - 1/3)/2)} + \frac{\sin(3\pi(u_x + 1/3))}{\sin(\pi(u_x + 1/3)/2)} \right) + 0.04 \left(\frac{\sin(3\pi(u_x - 2/3))}{\sin(\pi(u_x - 2/3)/2)} + \frac{\sin(3\pi(u_x + 2/3))}{\sin(\pi(u_x + 2/3)/2)} \right) \quad \dots(24)$$

This pattern is plotted using MATLAB in Figure 8 and Figure 9

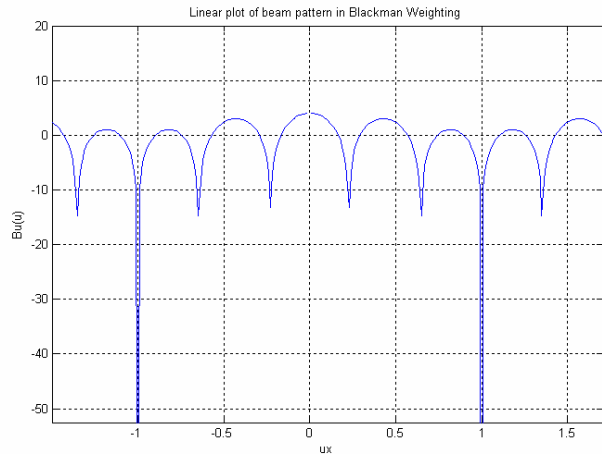


Figure 8

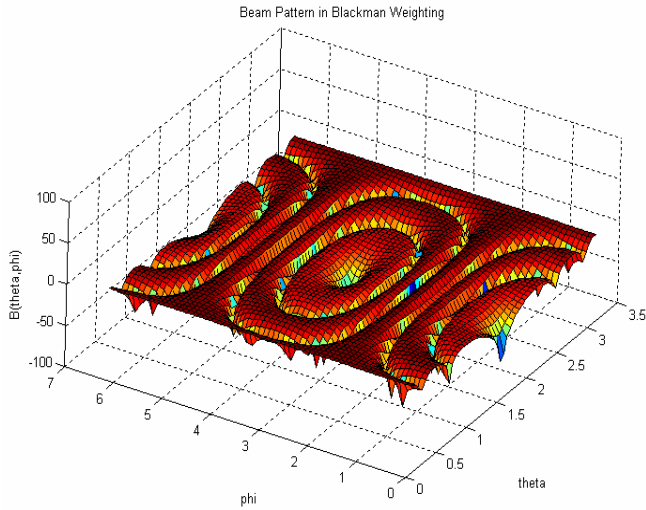


Figure 9

The directivity is given by
 $D = 0.577 * 6 = 3.462$ (25)

which is an improvement over all the previous weightings. The boresight (in u-space) is

$$BWFN = 6 * 2 / 6 = 2.0 \quad \dots (26)$$

E. Kaiser Weighting

Kaiser gave the formula for weights as an approximation to the more general prolate spheroidal sequences. Thus, for $d_x = \lambda / 2$,

$$w_n = I_0 \left(\beta \sqrt{1 - \frac{(2n-5)^2}{25}} \right) / I_0(\beta) \quad \dots(27)$$

for $n = 0, 1, \dots, 5$. where $I_0(x)$ is the modified Bessel function of first kind and order zero and β is parameter that controls the trade-off between bore sight and the height of first sidelobe. β is usually greater than 1.

A closed-form expression for beam pattern does not exist for Kaiser weighting., we obtain an Interpolating Function that can be used directly in *Mathematica* for plotting purposes.

For the choices $\beta = 3$ and $\beta = 6$, the plots are as shown in Figure 10 and Figure 11.

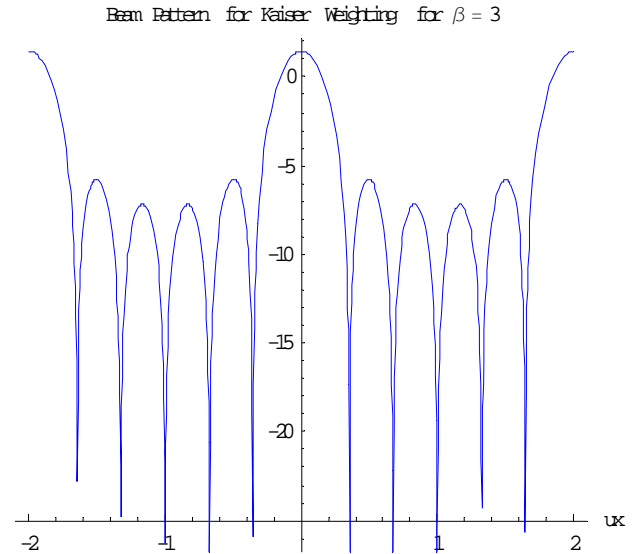


Figure 10

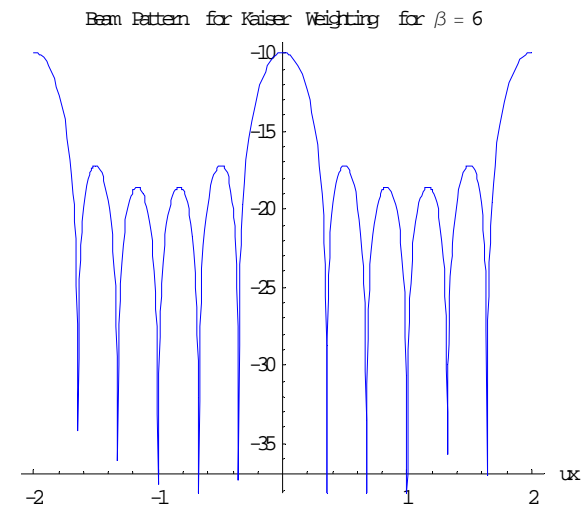


Figure 11

The directivity is obtained from the above plots as

$$D = 0.882 * 6 = 5.292 \quad \text{for } \beta = 3 \quad \dots (28)$$

$$D = 0.683 * 6 = 4.098 \quad \text{for } \beta = 6$$

The boresight (in u-space) is

$$\text{BWFN} = 1.75 \pi / 6 = 0.916 \quad \text{for } \beta = 3$$

$$\text{BWFN} = 2.76 \pi / 6 = 1.445 \quad \text{for } \beta = 6$$

....(29)

IV. THE 6 × 2 RECTANGULAR ARRAY

Consider a uniform rectangular array with six elements in a straight line along the x-axis and two elements along the y-axis as shown in fig 12

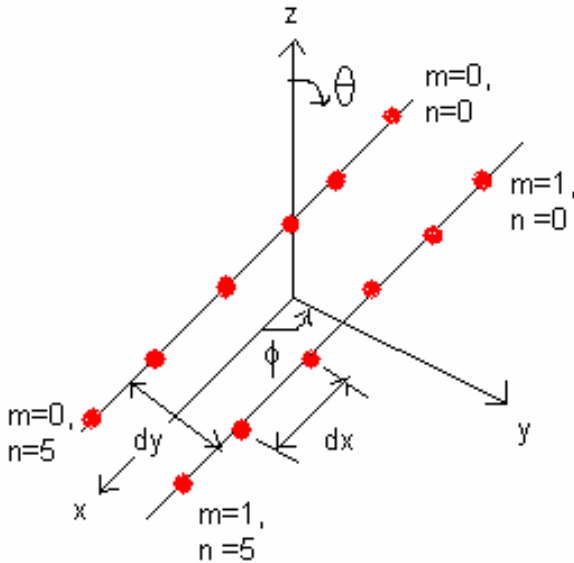


Figure 12

The distance between the elements along x-axis is d_x and between elements parallel to y-axis as d_y .

The 6×2 array manifold matrix for rectangular array is defined by:

$$\mathbf{V}_{\Psi}(\Psi) = \begin{pmatrix} 1 & e^{j\Psi_y} \\ e^{j\Psi_x} & e^{j(\Psi_x + \Psi_y)} \\ \vdots & \vdots \\ e^{j5\Psi_x} & e^{j(5\Psi_x + \Psi_y)} \end{pmatrix} \quad \dots(30)$$

where,

$$\Psi = \begin{pmatrix} \Psi_x \\ \Psi_y \end{pmatrix} \quad \dots(31)$$

$$\Psi_x = -k_x d_x = 2\pi u_x d_x / \lambda$$

$$\Psi_y = -k_y d_y = 2\pi u_y d_y / \lambda$$

The directional cosines are given by

$$u_x = \sin(\theta) \cos(\phi) \quad \dots(32)$$

$$u_y = \sin(\theta) \sin(\phi)$$

Similarly the 6×2 weight matrix is given by

$$\mathbf{W} = \begin{pmatrix} W_{00} & W_{01} \\ W_{10} & W_{11} \\ \vdots & \vdots \\ W_{50} & W_{51} \end{pmatrix} \quad \dots(33)$$

Analogous to 1-Dimensional array, the beam pattern for 6×2 array is

$$B(\Psi) = B(\Psi_x, \Psi_y) = \mathbf{W}^H \mathbf{V}_{\Psi}(\Psi)$$

$$= e^{-j(\frac{5}{2}\Psi_x + \frac{1}{2}\Psi_y)} \sum_{n=0}^5 (w_{n0}^* e^{jn\Psi_x} + w_{n1}^* e^{jn\Psi_x + \Psi_y})$$

....(34)

using the above equations, we obtain the beam pattern in terms of (θ, ϕ) .

V. BEAM PATTERNS OF THE 6 × 2 ARRAY

As we did in section III, here too we consider the same spectral weightings applied to the 6×2 array. For such weightings, we use the principle of *Pattern Multiplication*

$$W_{nm} = W_n \cdot W_m \quad \dots(35)$$

$$B(\Psi_x, \Psi_y) = B(\Psi_x) B(\Psi_y) \quad \dots(36)$$

To characterize beamwidth, two planes are chosen. The first is the elevation plane corresponding to $\phi = \phi_0$ and the second is a plane perpendicular to it, $\theta = \theta_0$. We call (θ_0, ϕ_0) as the *main response axis (MRA)* or *the steering direction*. Since we are considering 2-Dimensional arrays, instead of defining BWFN, a more complete picture is obtained by considering the *beam solid angle* Ω defined as

$$\Omega = \frac{\theta_{x0} \theta_{y0} \text{Sec } \theta_0}{\left(\text{Sin}^2 \phi_0 + \frac{\theta_{y0}^2}{\theta_{x0}^2} \text{Cos}^2 \phi_0 \right)^{1/2} \left(\text{Sin}^2 \phi_0 + \frac{\theta_{x0}^2}{\theta_{y0}^2} \text{Cos}^2 \phi_0 \right)^{1/2}} \quad \dots(37)$$

For the weightings that are considered below, $W_n = W_m$ and all weights are real. The directivity is again, given by eq (6), repeated here for convenience:

$$D = \left\{ \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi |B(\theta, \phi)|^2 \text{Sin } \theta \, d\theta \, d\phi \right\}^{-1} \quad \dots(38)$$

We shall integrate numerically to obtain the directivity.

A. Uniform Weighting

The beam pattern (for $d_x = \lambda/2$ and $d_y = \lambda/2$) is obtained by combining eq(8), eq(9), eq(32) and eq(33) :

$$B(u_x, u_y) = \frac{1}{6} \frac{\text{Sin}(3\pi u_x)}{\text{Sin}(\pi u_x/2)} \frac{1}{2} \frac{\text{Sin}(\pi u_y)}{\text{Sin}(\pi u_y/2)} \quad \dots(39)$$

A dB-plot of the above pattern is given in Figure 13.

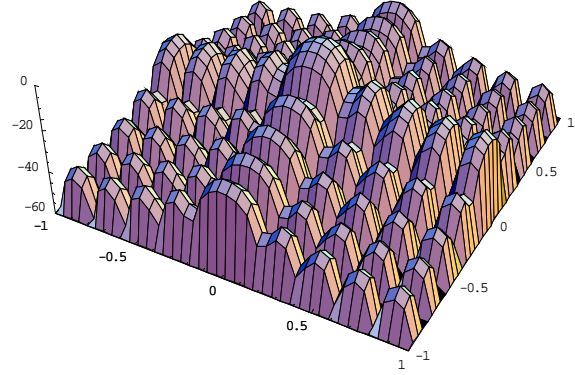


Figure 13

B. Hamming Weighting

The beam pattern (for $d_x = \lambda/2$ and $d_y = \lambda/2$) is obtained by combining eq(15), eq(16), eq(32) and eq(33) :

$$B_u(u_x) = 0.54 \frac{\text{Sin}(3\pi u_x)}{\text{Sin}(\pi u_x/2)} - 0.23 \left(\frac{\text{Sin}(3\pi(u_x - 1/3))}{\text{Sin}(\pi(u_x - 1/3)/2)} + \frac{\text{Sin}(3\pi(u_x + 1/3))}{\text{Sin}(\pi(u_x + 1/3)/2)} \right)$$

$$B_u(u_y) = 0.54 \frac{\text{Sin}(3\pi u_y)}{\text{Sin}(\pi u_y/2)} - 0.23 \left(\frac{\text{Sin}(3\pi(u_y - 1/3))}{\text{Sin}(\pi(u_y - 1/3)/2)} + \frac{\text{Sin}(3\pi(u_y + 1/3))}{\text{Sin}(\pi(u_y + 1/3)/2)} \right)$$

$$B_u(u_x, u_y) = B_u(u_x) B_u(u_y) \quad \dots(40)$$

A dB-plot of the above pattern is given in Figure 14.

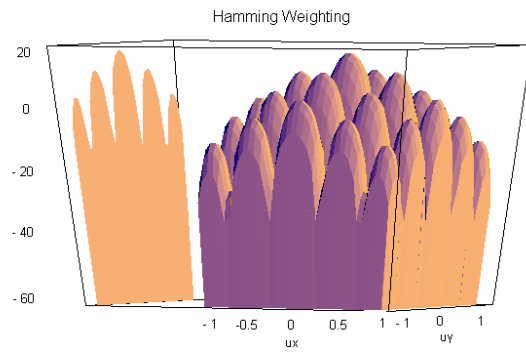


Figure 14

C. Hann Weighting

The beam pattern (for $d_x = \lambda / 2$ and $d_y = \lambda / 2$) is obtained by

$$B_u(u_x) = 0.5 \frac{\sin(3\pi u_x)}{\sin(\pi u_x / 2)} - 0.25 \left(\frac{\sin(3\pi(u_x - 1/3))}{\sin(\pi(u_x - 1/3)/2)} + \frac{\sin(3\pi(u_x + 1/3))}{\sin(\pi(u_x + 1/3)/2)} \right)$$

$$B_u(u_y) = 0.5 \frac{\sin(3\pi u_y)}{\sin(\pi u_y / 2)} - 0.25 \left(\frac{\sin(3\pi(u_y - 1/3))}{\sin(\pi(u_y - 1/3)/2)} + \frac{\sin(3\pi(u_y + 1/3))}{\sin(\pi(u_y + 1/3)/2)} \right)$$

$$B_u(u_x, u_y) = B_u(u_x) B_u(u_y) \dots(41)$$

A dB-plot of the above pattern is given in Figure 15.

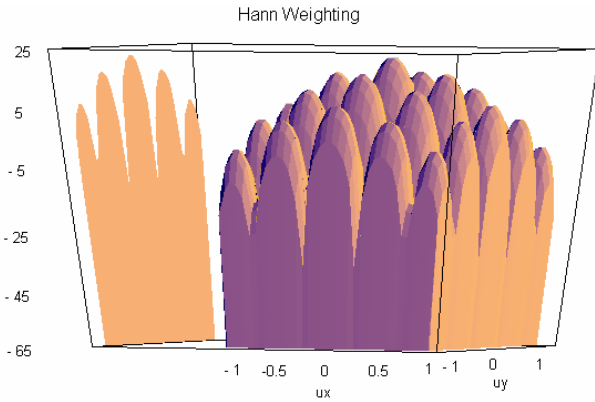


Figure 15

D. Blackman Weighting

The beam pattern (for $d_x = \lambda / 2$ and $d_y = \lambda / 2$) is obtained by

$$B_u(u_x) = 0.42 \frac{\sin(3\pi u_x)}{\sin(\pi u_x / 2)} - 0.25 \left(\frac{\sin(3\pi(u_x - 1/3))}{\sin(\pi(u_x - 1/3)/2)} + \frac{\sin(3\pi(u_x + 1/3))}{\sin(\pi(u_x + 1/3)/2)} \right) + 0.04 \left(\frac{\sin(3\pi(u_x - 2/3))}{\sin(\pi(u_x - 2/3)/2)} + \frac{\sin(3\pi(u_x + 2/3))}{\sin(\pi(u_x + 2/3)/2)} \right)$$

$$B_u(u_x, u_y) = B_u(u_x) B_u(u_y) \dots(42)$$

A dB-plot of the above pattern is given in Figure 16.

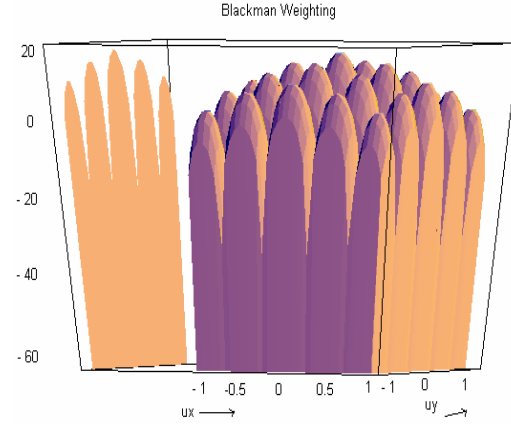


Figure 16

E. Kaiser Weighting

The beam pattern of Kaiser Weighting is given by:

$$B_u(u_x, u_y) = \left(e^{-j \frac{5}{2} \pi u_x} \sum_{n=0}^5 w_n(\beta) e^{jn \pi u_x} \right) \left(e^{-j \frac{1}{2} \pi u_y} \sum_{m=0}^1 w_m(\beta) e^{jm \pi u_y} \right)$$

where

$$w_k(\beta) = I_0 \left(\beta \sqrt{1 - \frac{(2k-5)^2}{25}} \right) / I_0(\beta) \quad \text{for } k = n, m$$

....(43)

The pattern is plotted in Figure 17

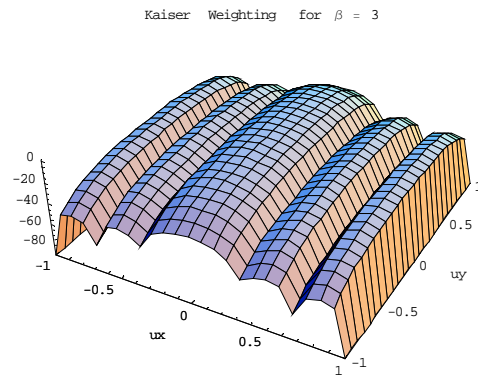


Figure 17

VI. CONCLUSIONS

We shall compare the different weighting functions in terms of three parameters which are important from a practical point of view.

The following results are observed for a 6×1 linear array (see Figure 1) with uniform spacing of $\lambda/2$.

(a) Directivity	
Uniform Weighting :	6.0
Hamming Weighting :	3.984
Hann Weighting :	4.002
Blackman Weighting :	3.462
Kaiser Weighting :	4.098
	for $\beta = 6$

Smaller the directivity, the narrower the mainlobe. In this sense, Blackman weighting is the best.

(b) BeamWidth between First Null (BWFN) or boresight in the u-space	
Uniform Weighting :	0.666
Hamming Weighting :	1.333
Hann Weighting :	1.333
Blackman Weighting :	2.0
Kaiser Weighting :	1.445
	for $\beta = 6$

Boresight is the smallest for Uniform Weighting, but it is the least effective with respect to sidelobe level (see below). Hence, practically Hamming weighting is a better choice.

(c) Height of the first sidelobe with respect to mainlobe (in dB)	
Uniform Weighting :	- 17.9
Hamming Weighting :	- 39.5
Hann Weighting :	- 31.4
Blackman Weighting :	- 56.6
Kaiser Weighting :	- 44.4
	for $\beta = 6$

Sidelobe level is the least for Blackman weighting, confirming its usefulness. Also, in (a) , (b) and (c) we see that Kaiser Weighting is the next best to Blackman weighting. Its control parameter β is not available with other methods. Thus Kaiser weighting is well-suited for adaptive systems.

For a 6×2 array (Figure 12) with a uniform spacing of $\lambda/2$ in all directions, we observe that the sidelobe level is comparatively low in Blackman weighting (Figure 16). However, Kaiser Weighting with $\beta = 3$ has a peculiar pattern (Figure 17) that does not change rapidly with the transverse direction (u_y). This may be very much needed in certain applications.

VII. REFERENCES

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