FIR FILTERS IN THE DESIGN OF ANTENNA BEAM PATTERNS

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ABSTRACT

An FIR (Finite Impulse Response) filter is, as the name itself signifies, a filter (frequency-selective device) whose impulse response in the time-domain is finite and has a countable number of filter coefficients. When the concept of filters is applied to antennas we find that the antenna weights are themselves the filter coefficients. Hence the vast body of results available for FIR filters can be used to characterize the spatial-frequency response of antennas.

An *antenna array* is any collection of isotropic point sources arranged according to a specified geometry. A *linear array* is an antenna array where the antenna elements (i.e. the point sources) are arranged in a straight line.It is a 1-dimensional array. A *planar array* is an antenna array arranged in a plane in some particular way. It is a 2-dimensional array. Common examples are rectangular arrays , circular arrays and elliptical arrays.

In this paper we are considering a linear array with 6 elements (called the 6 × 1 array) and a rectangular array with dimensions 6 × 2. Also , for simplicity , the distance between any two elements in any direction is assumed to be constant and equal.

Here we are concentrating on obtaining the beam patterns of these antenna arrays and measuring important parameters like mainlobe beamwidth (called bore sight) and directivity for different choices of weights . The analytical expressions for beam patterns are plotted and confirmed independently using MATLAB 7.0.1 and *Mathematica* 5.1

THE 6 × 1 LINEAR ARRAY

Consider a uniform linear array with six elements arranged in a straight line along the x-axis of the coordinate system as shown below :

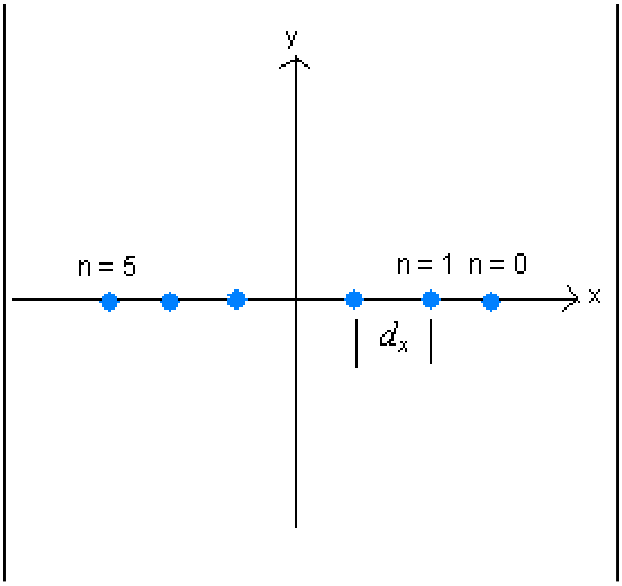

Figure 1

The inter-element distance is d_x.The weights of the antenna elements is given by the vector

$$\mathbf{w} = (w_0 \;\; w_1 \;\; w_2 \;\; w_3 \;\; w_4 \;\; w_5)^T \tag{1}$$

which is nothing but the impulse response of the FIR filter H(z)

$$H(z) = \sum_{n=0}^{5} w_n z^{-n} \tag{2}$$

The impulse response evaluated on the unit circle is obtained by substituting $z = \text{Exp}(j\,\psi)$ where ψ is the spatial frequency given by

$$\begin{aligned} \psi &= -k_x d_x \\ k_x &= \frac{2\pi}{\lambda} u_x \\ u_x &= \sin(\theta)\cos(\phi) \end{aligned} \tag{3}$$

where λ is the operating wavelength, k_x is the wavenumber along x-axis, u_x is the unit vector along x-axis in the spherical coordinates (r,θ,ϕ) .The distance between any two elements is taken to be d_x. The x-coordinate of n^{th} element , x_n , is given by

$$x_n = \left(n - \frac{5}{2}\right) d_x \qquad /;\ n = 0, 1, \ldots, 5 \tag{4}$$

The array manifold vector (denoted in bold-face) is defined by

$$\mathbf{v}(k_x) = \left(e^{j\frac{5}{2}k_x d_x} \quad e^{j\frac{3}{2}k_x d_x} \quad .. \quad e^{j\frac{-3}{2}k_x d_x} \quad e^{j\frac{-5}{2}k_x d_x} \right)^T \tag{5}$$

The beam pattern in the u-space is given by

$$B_u(u) = w^H v(u) = e^{-j\frac{5}{2}2\pi d_x u_x/\lambda} \sum_{n=0}^{5} w_n e^{j n 2\pi d_x u_x/\lambda} \tag{6}$$

Similarly the actual beam pattern in 3-D space (i.e. in terms of θ and ϕ) is obtained by substituting the expression for u_x from *eq(3)* into *eq(6)*.

BEAM PATTERNS OF THE 6 × 1 ARRAY

The different choices of **w** in *eq(1)* gives a plethora of beam patterns that can be applied practically. Here we shall specifically concentrate on three important weighting functions :

(a) Uniform weighting
(b) Hamming weighting
(c) Hann weighting
(d) Blackman weighting
(e) Kaiser weighting

We compare these weighting functions with respect to crucial parameters such as directivity and boresight (i.e. width of the mainlobe).

The directivity is given by

$$D = \left(\frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \mathrm{abs}[B(\theta, \phi)]^2 \sin(\theta)\, d\theta\, d\phi \right)^{-1} \tag{7}$$

When $d_x = \lambda / 2$, we call the array as a *standard linear array*. For such an array, the directivity is (see Ref[1] , pp. 60-62)

$$D = \left(\sum_{n=0}^{5} w_n^2 \right)^{-1} \tag{8}$$

Practically speaking, *eq(7)* has a closed-form expression only in a few cases. Hence in general , we perform numerical integration to obtain directivity.

Uniform Weighting

The weights are given by

$$w_n = \frac{1}{6} \quad /;\ n = 0, 1, \ ...5 \tag{9}$$

Substituting this in *eq(6)* and performing the summation in *Mathematica* (using the command **Sum**) gives the *normalized* beam pattern as:

$$B_u\,(u) = \frac{1}{6}\ \frac{\sin\,(6\,\pi\,d_x\,u_x\,/\,\lambda)}{\sin\,(\pi\,d_x\,u_x\ /\,\lambda)} \tag{10}$$

For a standard linear array $d_x = \lambda\,/\,2$. Hence

$$B_u\,(u) = \frac{1}{6}\ \frac{\sin\,(3\,\pi\,u_x)}{\sin\,(\pi\,u_x\ /\,2)} \tag{11}$$

This pattern is plotted in Figure 2 (using MATLAB) as a function of u_x and in Figure 3 (using *Mathematica*) as a function of θ and ϕ.

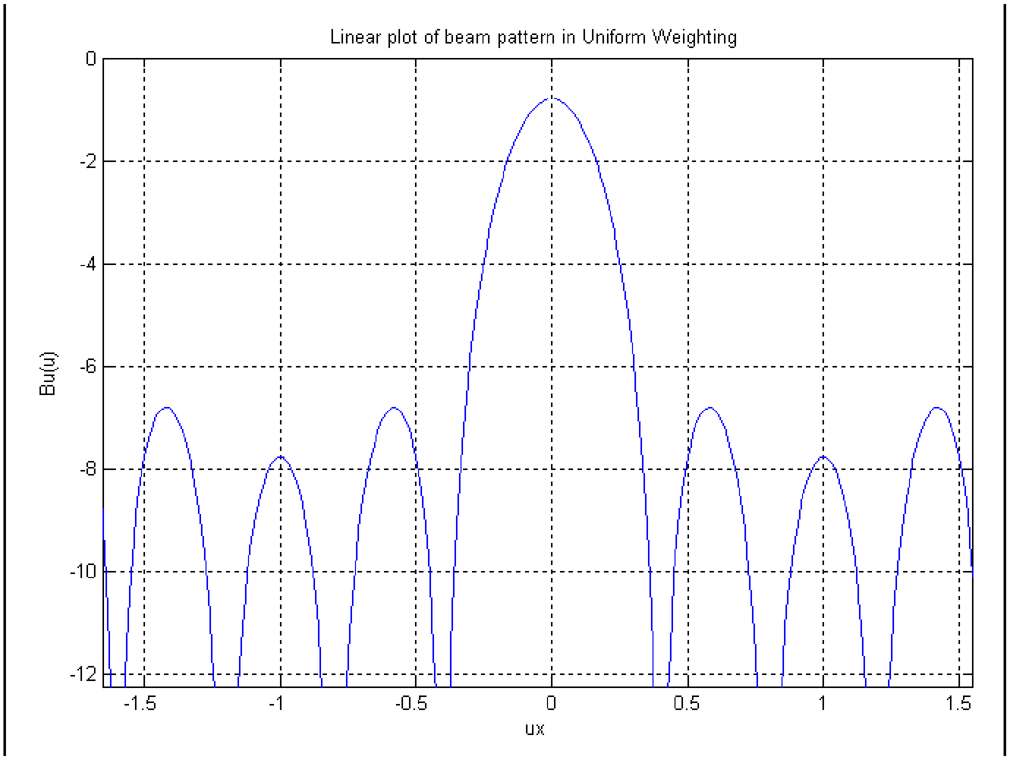

Figure 2

The boresight (BeamWidth between First Null - BWFN) in the u-space is given by :

$$\text{BWFN} = 2\,\lambda\ /\ (6\,d_x)\ = 0.666 \quad \text{when}\ d_x = \lambda\,/\,2 \tag{12}$$

The directivity D is given by (Ref[1] , eq. 2.166) ,

$$D = 4 / \mathrm{BWFN} = 6 \qquad \text{when } d_x = \lambda / 2 \tag{13}$$

The same result is obtained by using *eq(11)* in *eq(8)* simplifying the summation.

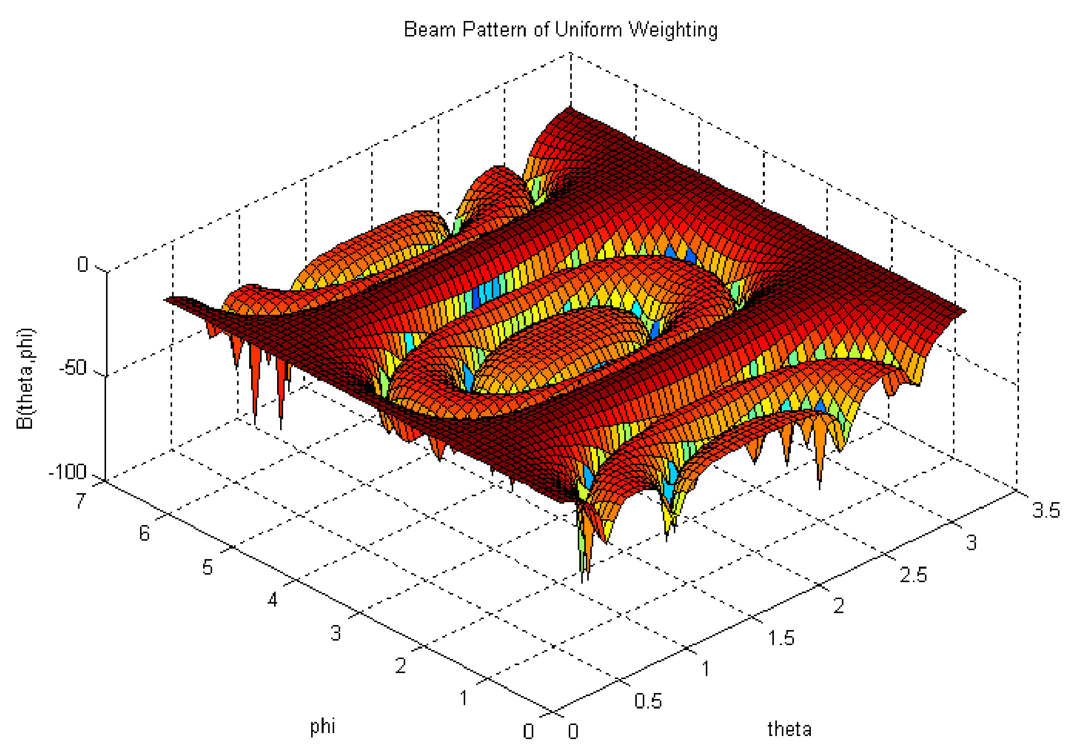

Figure 3

■ Hamming Weighting

In Hamming weighting we place a null at $u_x = 3 / 6 = 1 / 2$ and normalize the response at broadside to unity. Thus ,

$$w_n \equiv w(n) = 0.54 - 0.46 \cos\left(\frac{2 \pi n}{5}\right) \qquad /;\ n = 0,\ 1.\,.\,.\,5 \tag{14}$$

Performing the summation (with $d_x = \lambda / 2$) in *Mathematica* (using the command **Sum**) in *eq(8)* gives

$$B_u(u) = 0.54 \frac{\sin(3 \pi u_x)}{\sin(\pi u_x / 2)} - 0.23 \left(\frac{\sin(3 \pi (u_x - 1/3))}{\sin(\pi (u_x - 1/3) / 2)} + \frac{\sin(3 \pi (u_x + 1/3))}{\sin(\pi (u_x + 1/3) / 2)}\right) \tag{15}$$

This pattern is plotted in Figure 4 and Figure 5

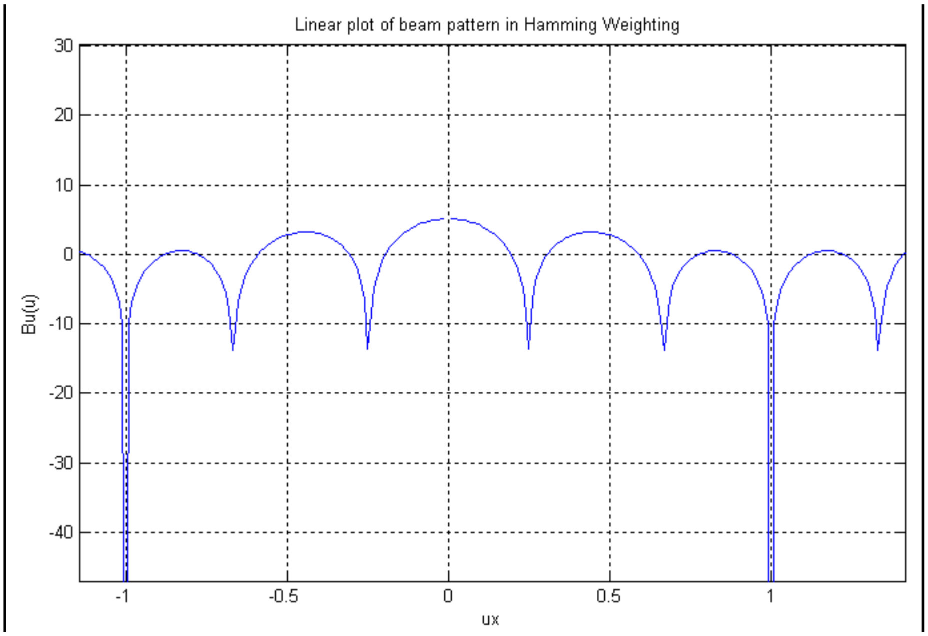

Figure 4

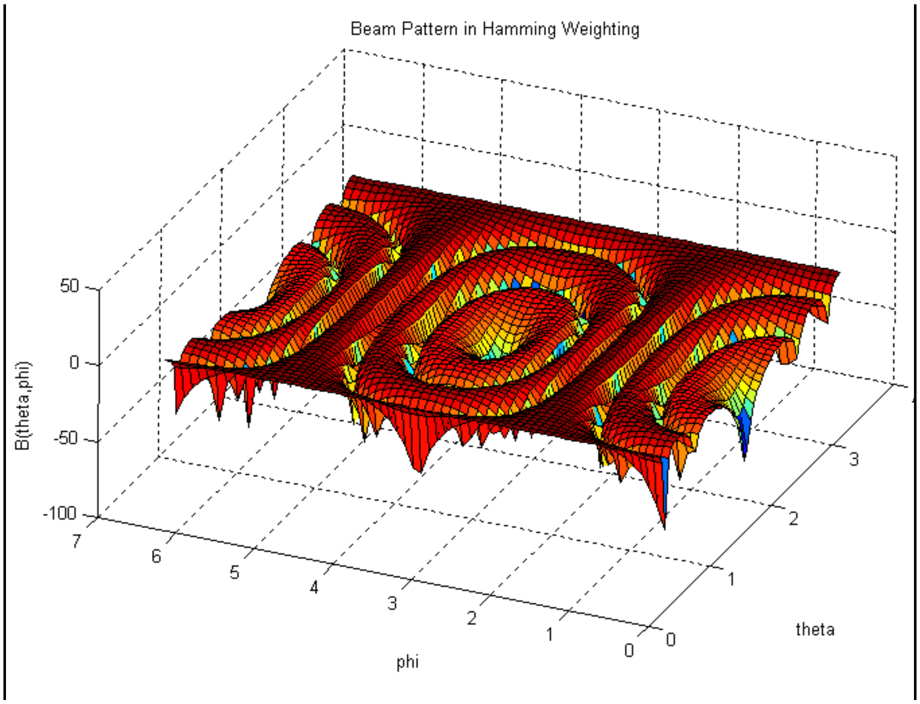

Figure 5

The directivity is obtained by substituting eq(15) in eq(7) and integrating numerically (using the *Quasi-Monte Carlo* option in the command **NIntegrate** of *Mathematica*) :

$$\begin{gathered} D = 3.984 \\ \mathrm{BWFN} = 4/3 = 1.333 \end{gathered} \tag{16}$$

Hann Weighting

The weights are given by

$$w_n \equiv w(n) = 0.5 - 0.5\cos\left(\frac{2\pi n}{5}\right) \quad /;\ n = 0,\ 1. . . 5 \tag{17}$$

Performing the summation (with $d_x = \lambda / 2$) in *Mathematica* (using the command **Sum**) in *eq(8)* gives

$$\begin{gathered} B_u(u) = 0.5\frac{\sin(3\pi u_x)}{\sin(\pi u_x/2)} - \\ 0.25\left(\frac{\sin(3\pi(u_x - 1/3))}{\sin(\pi(u_x - 1/3)/2)} + \frac{\sin(3\pi(u_x + 1/3))}{\sin(\pi(u_x + 1/3)/2)}\right) \end{gathered} \tag{18}$$

This pattern is plotted in Figure 4 and Figure 5

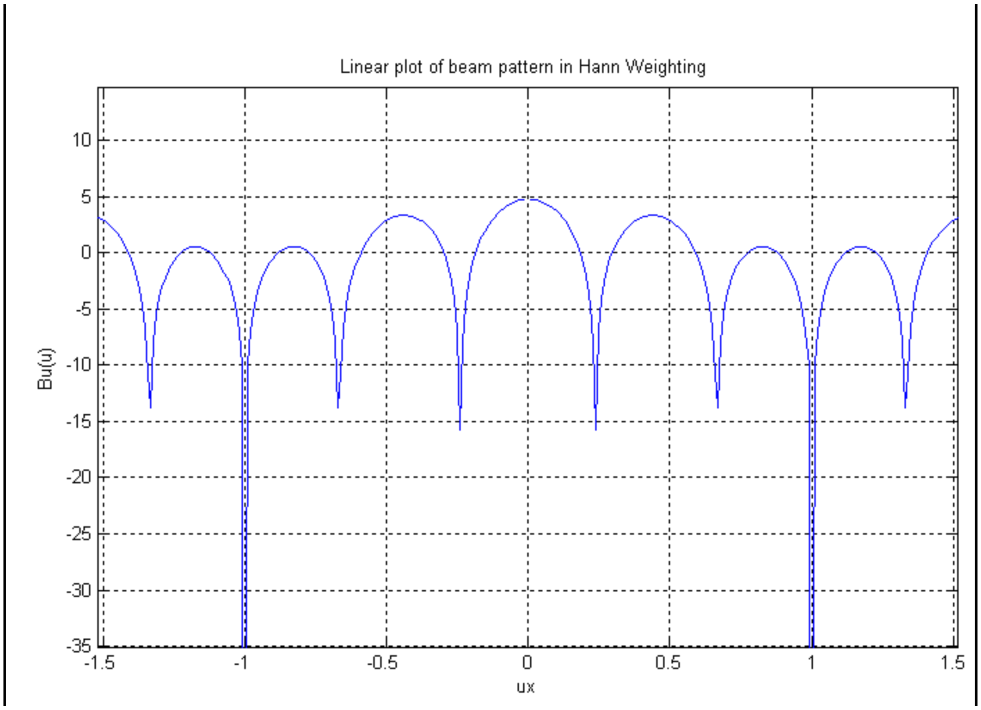

Figure 6

The directivity is obtained by substituting eq(18) in eq(7) and integrating numerically (using the *GaussKronrod* option in the command **NIntegrate** of *Mathematica*) :

$$\begin{gathered} D = 4.002 \\ \mathrm{BWFN} = 1.3333 \end{gathered} \tag{19}$$

which is same as that of Hamming Weigthing.

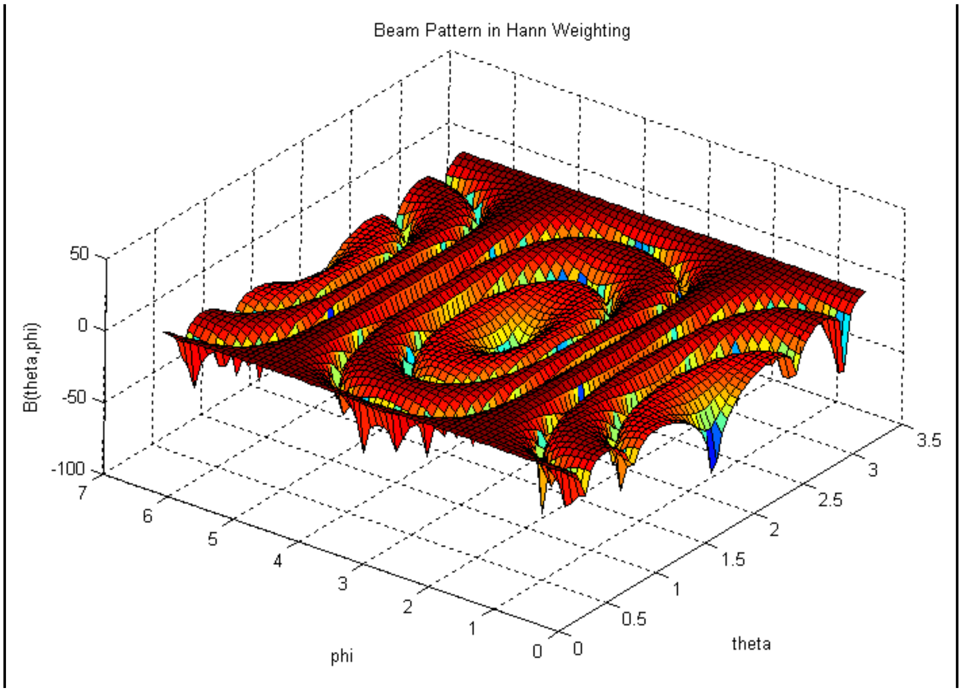

Figure 7

■ Blackman Weighting

Blackman proposed the following formula for weights so as to produce nulls at the peaks of first two sidelobes.

$$w_n = 0.42 - 0.5\cos\left(\frac{2\pi n}{5}\right) + 0.08\cos\left(\frac{4\pi n}{5}\right) \quad /;\ n = 0,\ 1,\ ..\ 5 \qquad \textbf{(20)}$$

Performing the summation (with $d_x = \lambda / 2$) in *Mathematica* (using the command **Sum**) in *eq(8)* gives

$$\begin{aligned} B_u(u) = {} & 0.42\,\frac{\sin(3\pi u_x)}{\sin(\pi u_x/2)} - \\ & 0.25\left(\frac{\sin(3\pi(u_x-1/3))}{\sin(\pi(u_x-1/3)/2)} + \frac{\sin(3\pi(u_x+1/3))}{\sin(\pi(u_x+1/3)/2)}\right) + \qquad \textbf{(21)} \\ & 0.04\left(\frac{\sin(3\pi(u_x-2/3))}{\sin(\pi(u_x-2/3)/2)} + \frac{\sin(3\pi(u_x+2/3))}{\sin(\pi(u_x+2/3)/2)}\right) \end{aligned}$$

This pattern is plotted in Figure 8 and Figure 9

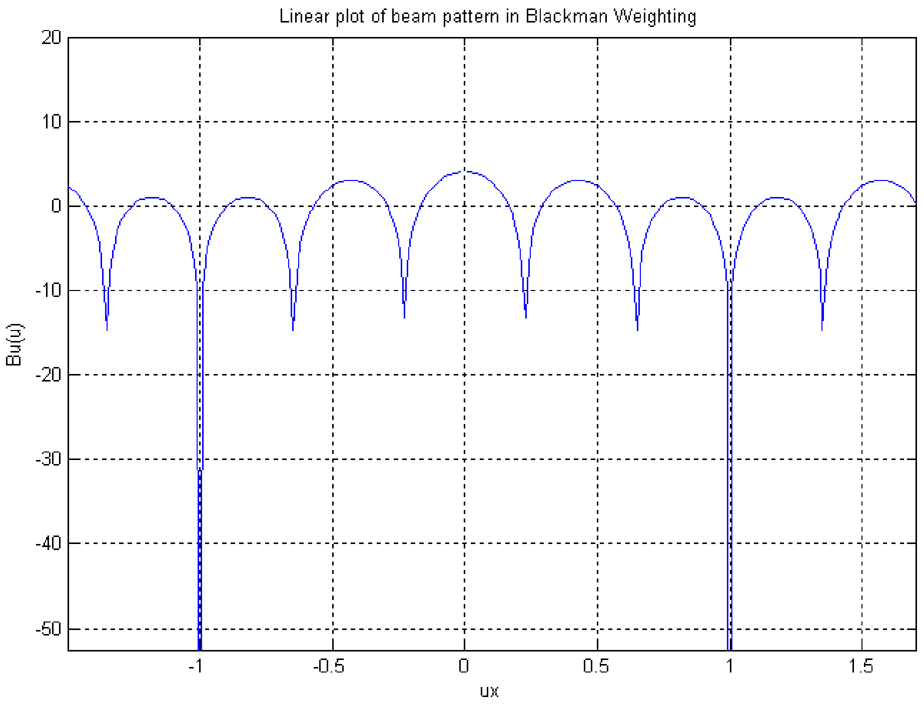

Figure 8

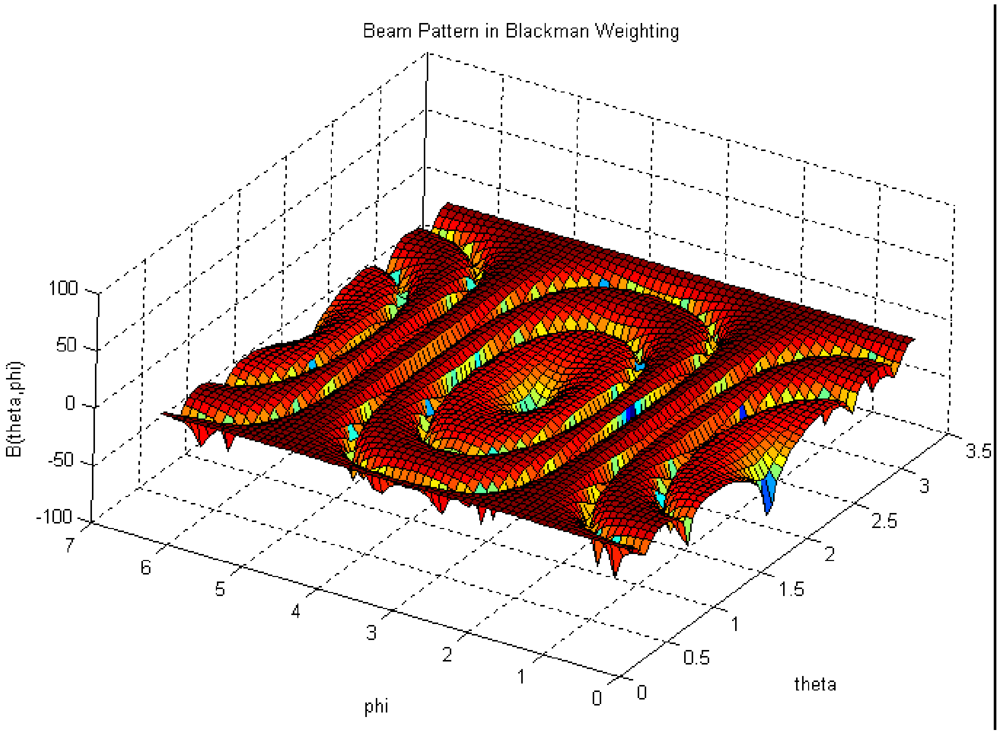

Figure 9

The directivity is obtained by substituting eq(21) in eq(7) and integrating numerically (using the MultiDimensional option in the command **NIntegrate** of Mathematica):

$$\begin{aligned} D &= 3.462 \\ \mathrm{BWFN} &= 2.00 \end{aligned} \tag{22}$$

■ Kaiser Weighting

Kaiser gave the formula for weights as an approximation to the more general prolate spheroidal sequences. Thus , for $d_x = \lambda / 2$,

$$w_n = I_0\left(\beta\sqrt{1-\frac{(2n-5)^2}{25}}\right) \Big/ I_0(\beta) \qquad /;\ n = 0, 1, \ ...5 \tag{23}$$

where $I_0(x)$ is the modified Bessel function of first kind and order zero (Ref[2]) and β is parameter that controls the trade-off between bore sight and the height of first sidelobe. β is usually greater than 1.

A closed-form expression for beam pattern does not exist for Kaiser weighting. However, using *eq(6)* and performing numerical summation (using the command **NSum** in *Mathematica*), we obtain an Interpolating Function that can be used directly in *Mathematica* for plotting purposes.

For the choices $\beta = 3$ and $\beta = 6$, the plots are as shown in Figure 10

The directivity and bore-sight in the u-space are obtained from the plots as :

$$\begin{aligned} D &= 5.292 \qquad /;\ \beta = 3 \\ D &= 4.098 \qquad /;\ \beta = 6 \\ \mathrm{BWFN} &= 0.916 \qquad /;\ \beta = 3 \\ \mathrm{BWFN} &= 1.445 \qquad /;\ \beta = 6 \end{aligned} \tag{24}$$

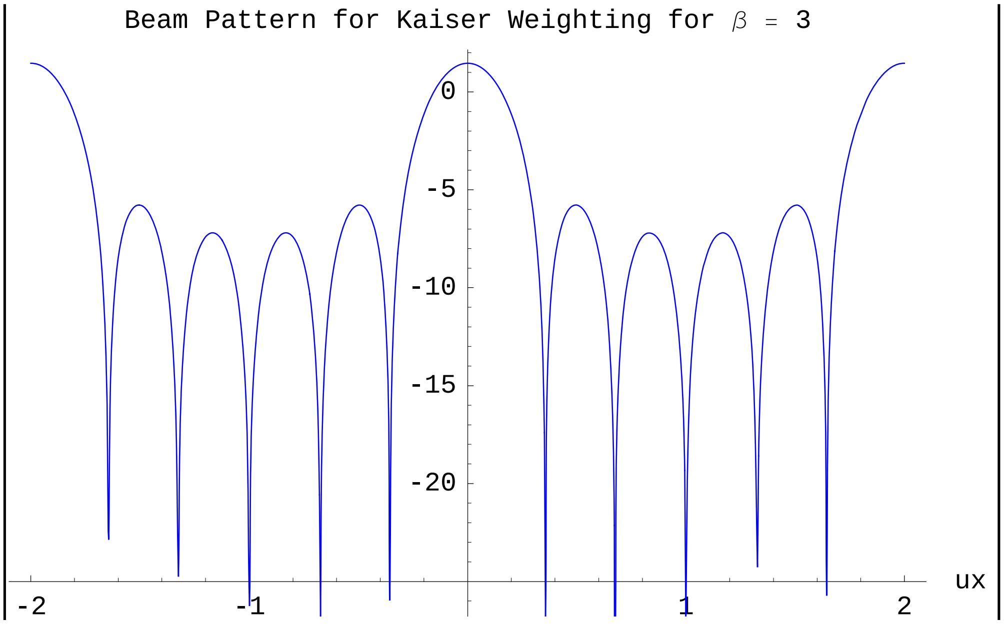

Figure 10

THE 6 × 2 RECTANGULAR ARRAY

Consider a uniform rectangular array with six elements in a straight line along the x-axis and two elements along the y-axis as shown below :

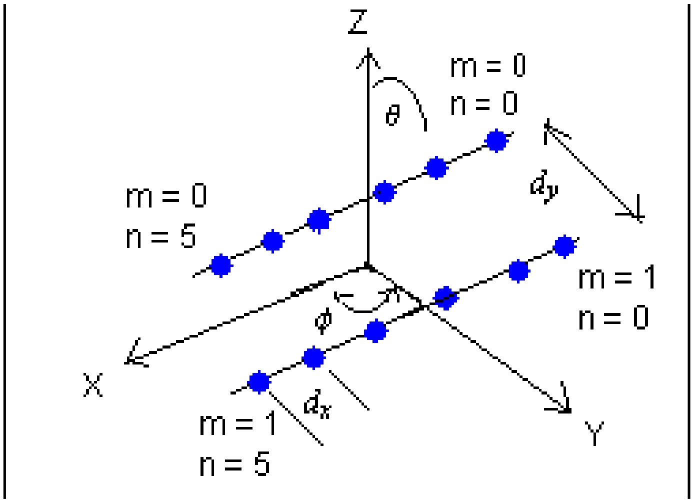

Figure 11

The distance between the elements along x-axis is d_x and between elements parallel to y-axis as d_y. The 6 × 2 array manifold matrix $\mathbf{V}(\Psi)$ for the rectangular array is defined as :

$$\mathbf{V}(\boldsymbol{\Psi}) = \begin{pmatrix} 1 & e^{j\Psi_x} & . & . & e^{j5\Psi_x} \\ e^{j\Psi_y} & e^{j(\Psi_x+\Psi_y)} & . & . & e^{j5(\Psi_x+\Psi_y)} \end{pmatrix}^T$$

$$\boldsymbol{\Psi} = \begin{pmatrix} \Psi_x & \Psi_y \end{pmatrix}^T$$

$$\Psi_x = -k_x d_x = 2\pi u_x d_x / \lambda$$

$$\Psi_y = -k_y d_y = 2\pi u_y d_y / \lambda$$

$$u_x = \sin(\Theta)\cos(\phi)$$

$$u_y = \sin(\Theta)\sin(\phi)$$

where $\boldsymbol{\Psi}$ is the wavenumber with components Ψ_x and Ψ_y along the x- and y- axes respectively. The unit vectors along the x- and y-axes are given by u_x and u_y . The 6 × 2 weight matrix is

$$\mathbf{W} = \begin{pmatrix} w_{00} & w_{10} & . & . & w_{50} \\ w_{01} & w_{11} & . & . & w_{51} \end{pmatrix}^T \tag{26}$$

Analogous to 1-Dimensional array, the beam pattern for 6 × 2 array is

$$\begin{aligned} B(\boldsymbol{\Psi}) &= B(\Psi_x, \Psi_y) = \mathbf{W}^T \mathbf{V}(\boldsymbol{\Psi}) \\ &= e^{-j\left(\frac{5}{2}\Psi_x + \frac{1}{2}\Psi_y\right)} \sum_{n=0}^{5} \left(w_{n0}\, e^{j n \Psi_x} + w_{n1}\, e^{j n (\Psi_x + \Psi_y)}\right) \end{aligned} \tag{27}$$

BEAM PATTERNS OF THE 6 × 2 RECTANGULAR ARRAY

Here we consider the same spectral weightings applied to the 6 × 2 array. For such weightings, we use the principle of *Pattern Multiplication* (Ref[3]) to rewrite *eq(27)* as:

$$\begin{aligned} w_{n\,m} &= w_n\, w_m \\ B(\Psi_x, \Psi_y) &= B(\Psi_x)\, B(\Psi_y) \end{aligned} \tag{28}$$

For the weightings that are considered below , $w_n = w_m$ and all weights are real. The directivity is again given by *Eq(7)* .

■ Uniform Weighting

The beam pattern (for $d_x = \lambda / 2$ and $d_y = \lambda/2$) is obtained by combining *eq(9)* , *eq(11)* and *eq(28):*

$$B(u_x, u_y) = \frac{1}{6}\,\frac{\sin(3\pi u_x)}{\sin(\pi u_x / 2)}\;\frac{1}{2}\,\frac{\sin(\pi u_y)}{\sin(\pi u_y / 2)} \tag{29}$$

A dB-plot of the pattern is given in Figure 12 :

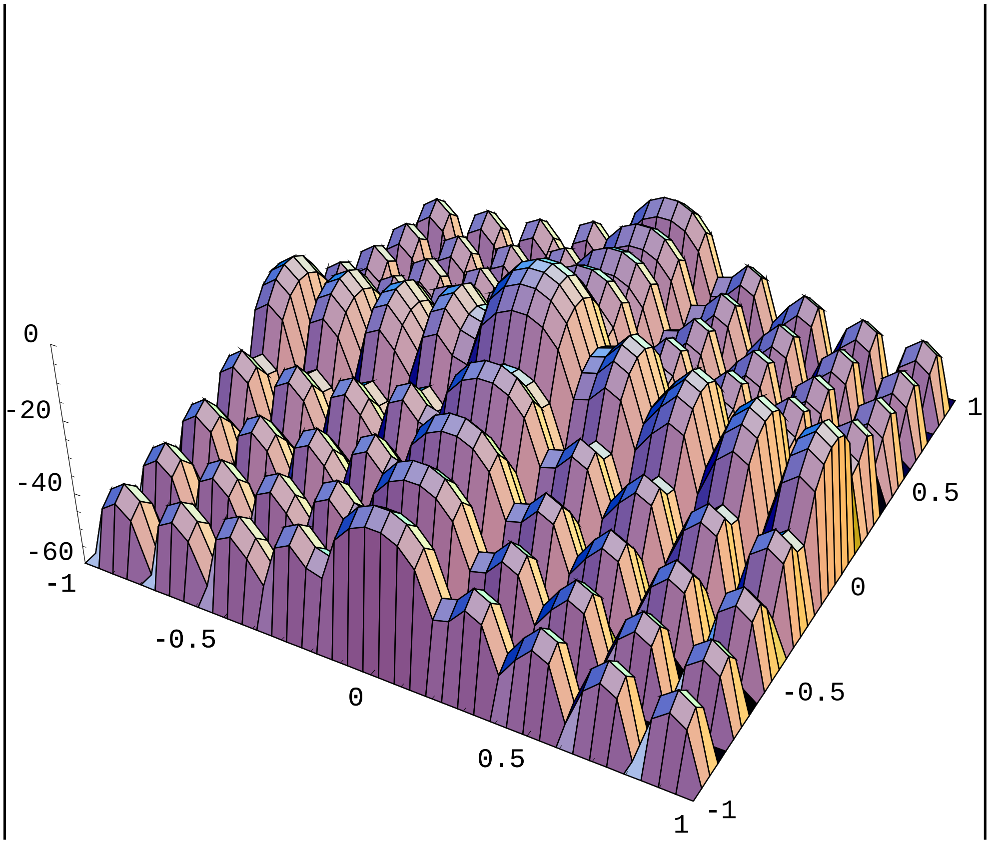

Figure 12

■ Hamming Weighting

The beam pattern (for $d_x = \lambda / 2$ and $d_y = \lambda/2$) is obtained by combining *eq(15)* and *eq(28)* :

$$
\begin{aligned}
B(u_x) &= 0.54\,\frac{\sin(3\pi u_x)}{\sin(\pi u_x/2)} - \\
&\quad 0.23\left(\frac{\sin(3\pi(u_x - 1/3))}{\sin(\pi(u_x - 1/3)/2)} + \frac{\sin(3\pi(u_x + 1/3))}{\sin(\pi(u_x + 1/3)/2)}\right) \\
B(u_y) &= 0.54\,\frac{\sin(3\pi u_y)}{\sin(\pi u_y/2)} - \\
&\quad 0.23\left(\frac{\sin(3\pi(u_y - 1/3))}{\sin(\pi(u_y - 1/3)/2)} + \frac{\sin(3\pi(u_y + 1/3))}{\sin(\pi(u_y + 1/3)/2)}\right) \\
B(u_x, u_y) &= B(u_x)\,B(u_y)
\end{aligned}
\qquad \textbf{(30)}
$$

A dB- plot of the above pattern is given in Figure 13

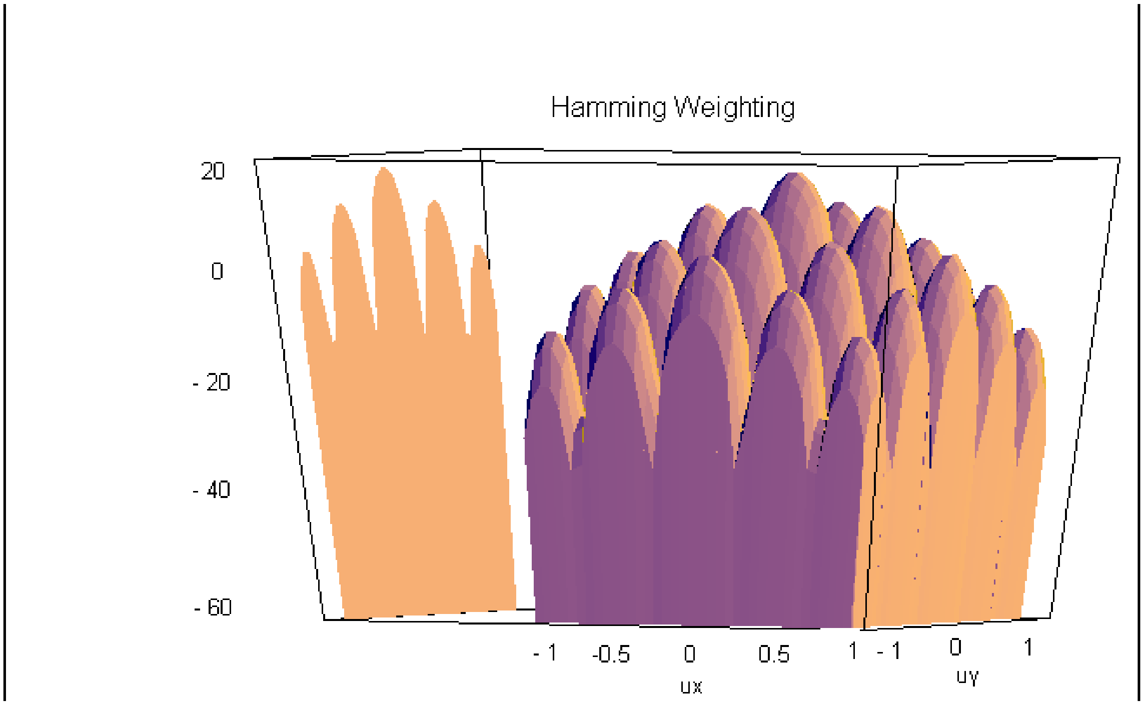

Figure 13

■ Hann Weighting

The beam pattern (for $d_x = \lambda / 2$ and $d_y = \lambda/2$) is obtained by combining *eq(18)* and *eq(28)* :

$$\begin{aligned} B(u_x) &= 0.5\,\frac{\sin(3\pi u_x)}{\sin(\pi u_x/2)} - \\ &\quad 0.25\left(\frac{\sin(3\pi(u_x - 1/3))}{\sin(\pi(u_x - 1/3)/2)} + \frac{\sin(3\pi(u_x + 1/3))}{\sin(\pi(u_x + 1/3)/2)}\right) \\ B(u_y) &= 0.5\,\frac{\sin(3\pi u_y)}{\sin(\pi u_y/2)} - \\ &\quad 0.25\left(\frac{\sin(3\pi(u_y - 1/3))}{\sin(\pi(u_y - 1/3)/2)} + \frac{\sin(3\pi(u_y + 1/3))}{\sin(\pi(u_y + 1/3)/2)}\right) \\ B(u_x, u_y) &= B(u_x)\;B(u_y) \end{aligned} \tag{31}$$

A dB- plot of the above pattern is given in Figure 14

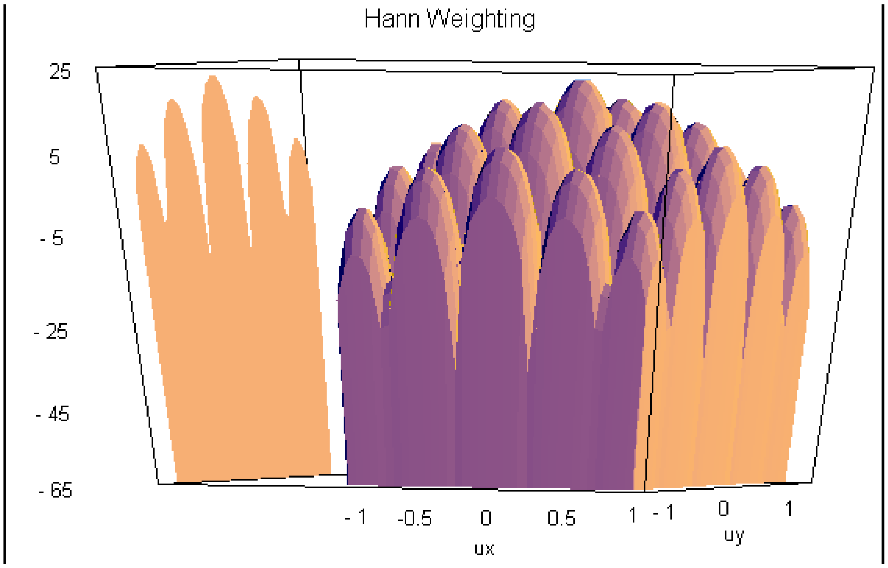

Figure 14

Blackman Weighting

The beam pattern (for $d_x = \lambda / 2$ and $d_y = \lambda/2$) is obtained by combining *eq(21)* and *eq(28)* :

$$B(u_x) = 0.42 \frac{\sin(3\pi u_x)}{\sin(\pi u_x / 2)} - 0.25 \left(\frac{\sin(3\pi (u_x - 1/3))}{\sin(\pi (u_x - 1/3) / 2)} + \frac{\sin(3\pi (u_x + 1/3))}{\sin(\pi (u_x + 1/3) / 2)} \right) + 0.04 \left(\frac{\sin(3\pi (u_x - 2/3))}{\sin(\pi (u_x - 2/3) / 2)} + \frac{\sin(3\pi (u_x + 2/3))}{\sin(\pi (u_x + 2/3) / 2)} \right) \qquad (32)$$

$B(u_y)$ is obtained by replacing u_x by u_y in the above expression. The total pattern is $B(u_x)\, B(u_y)$. A dB- plot of the above pattern is given in Figure 15

Kaiser Weighting

The beam pattern of Kaiser weighting (see *eq(23)* and *eq(27)*) is given by :

$$B(u_x, u_y) = \left(e^{-j \frac{5}{2} \pi u_x} \sum_{n=0}^{5} w_n(\beta)\, e^{j n \pi u_x} \right) \left(e^{-j \frac{1}{2} \pi u_y} \sum_{m=0}^{1} w_m(\beta)\, e^{j m \pi u_y} \right) \qquad (33)$$

where $w_k(\beta)$ for k=n,m is given by *Eq(23)* .The beam pattern using *Eq(33)* is plotted in Fig 16.

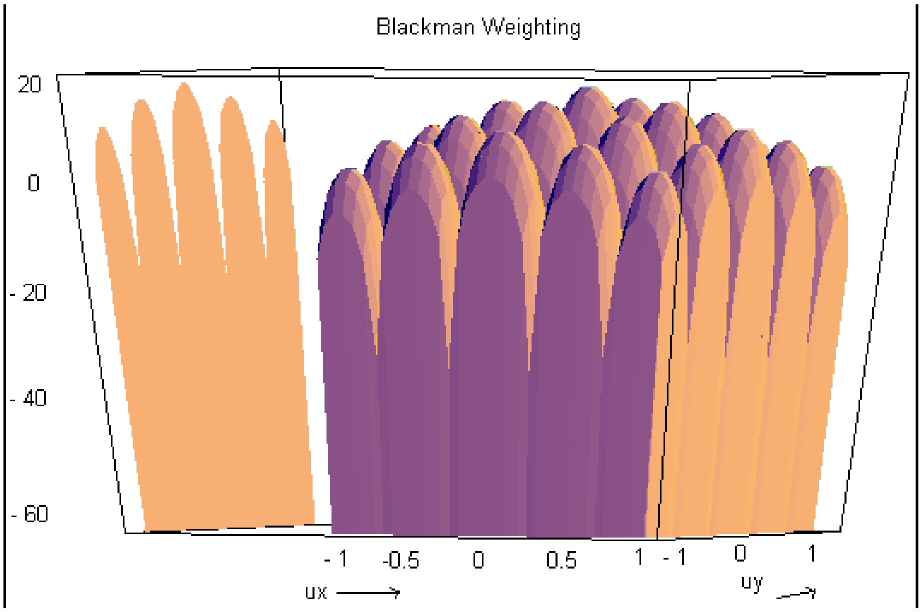

Figure 15

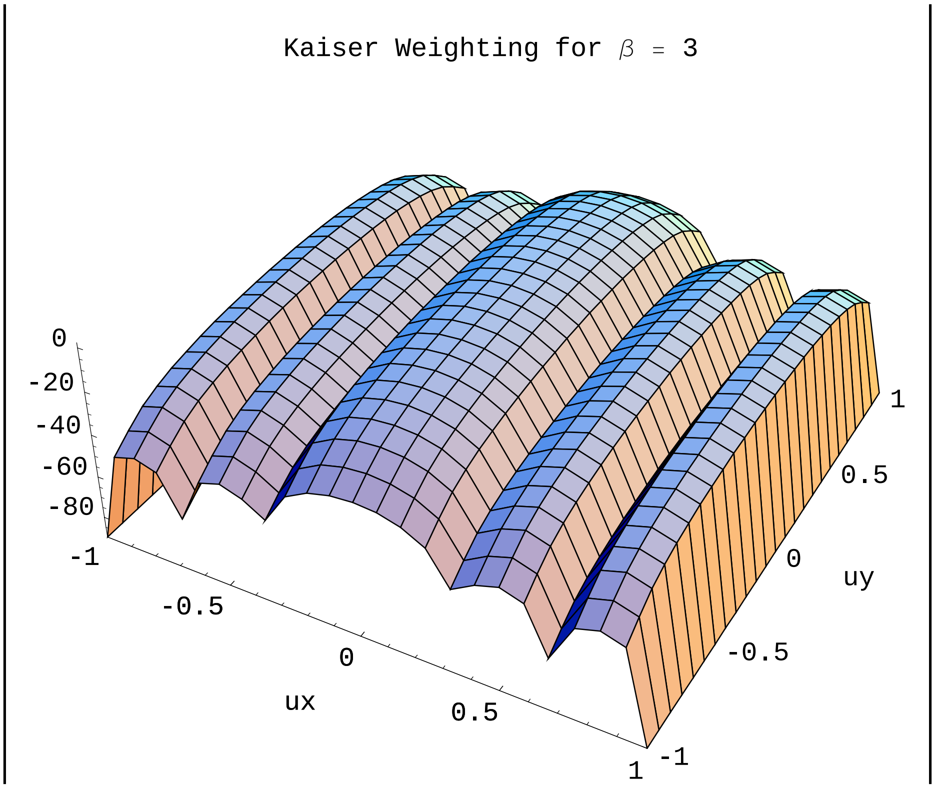

Figure 16

CONCLUSIONS

We shall compare the different weighting functions in terms of three parameters which are important from a practical point of view.

The following results are observed for a 6 × 1 linear array (see Figure 1) with uniform spacing of $\lambda/2$.

(a) Directivity

Uniform Weighting : 6.0
Hamming Weighting : 3.984
Hann Weighting : 4.002
Blackman Weighting : 3.462
Kaiser Weighting : 4.098 for $\beta = 6$

Smaller the directivity, the narrower the mainlobe. In this sense, Blackman weighting is the best.

(b) BeamWidth between First Null (BWFN) or boresight in the u-space

Uniform Weighting : 0.666
Hamming Weighting : 1.333
Hann Weighting : 1.333
Blackman Weighting : 2.0
Kaiser Weighting : 1.445 for $\beta = 6$

Boresight is the smallest for Uniform Weighting , but it is the least effective with respect to sidelobe level (see below). Hence , practically Hamming weighting is a better choice.

(c) Height of the first sidelobe with respect to mainlobe (in dB)

Uniform Weighting : - 17.9
Hamming Weighting : - 39.5
Hann Weighting : - 31.4
Blackman Weighting : - 56.6
Kaiser Weighting : - 44.4 for $\beta = 6$

Sidelobe level is the least for Blackman weighting, confirming its usefulness.Also, in (a) , (b) and (c) we see that Kaiser Weighting is the next best to Blackman weighting. Its control parameter β is not available with other methods. Thus Kaiser weighting is well-suited for adaptive systems.

For a 6 × 2 array (Figure 11) with a uniform spacing of $\lambda/2$ in all directions, we observe that the sidelobe level is comparatively low in Blackman weighting (Figure 15). However , Kaiser Weighting with $\beta = 3$ has a peculiar pattern (Figure 16) that does not change rapidly with the transverse direction (u_y). This may be very much needed in certain applications.

REFERENCES

[1] Harry L. Van Trees , *Optimum Array Processing* , Wiley Interscience , New York, 2002

[2] http://functions.wolfram.com/BesselAiryStruveFunctions/BesselI

[3] John.D.Kraus, *Antennas : For All Applications*, Tata Mc-Graw Hill ,3rd edition, 2002

[4] R.B.Blackman, *Linear Data-Smoothing and Prediction in Theory and Practice*. Addison-Wesley,Massachusetts, 1965

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The software code for all the figures in this paper can be obtained by requesting the author through bdsatish@gmail.com